

On identities in centrally nilpotent Moufang loops and centrally nilpotent A -loops

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Abstract

This paper proves that the variety generated by a centrally nilpotent Moufang loop (or centrally nilpotent A -loop) is finitely based.

Key words: Moufang loop, A -loop, centrally nilpotent loop, lower central series, commutator-associator of type (α, β) of weight n commutator-associators of type (μ) of weight n , fully invariant subloop, word subloop, basis of identities.

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Combining the techniques used by Lyndon [13] in showing that the identities of a nilpotent group are finitely based with some related ideas of Higman [11] and the results of Bruck [3] on the structure of commutative Moufang loops Evans [9] prove that the identities of centrally nilpotent commutative Moufang loop are finitely based. In this paper the Evans's result is reinforced for centrally nilpotent Moufang loops and centrally nilpotent A -loops (Theorem 5.4), following the ideas of [9].

In [2], [3] Bruck developed and investigated in details the theory of nilpotency, particularly centrally nilpotency, for loops on the basis of nilpotency for groups.

The theory of centrally nilpotent loops essentially differs from the theory of nilpotent groups. In some of his papers, Sandu offers methods to research centrally nilpotent loops, distinct from [2], [3]. They are systematized in [6] and [7]. The notions of commutators, of commutators of weight n , of lower central series, of upper central series etc. for groups are replaced by the notions of commutator-associator, of commutator-associators of weight n , of lower central series, of upper central series, etc. for loops respectively. The basic result of this paper, Theorem 5.4, is proved using the mentioned replacement. Particularly, it uses the structural construction of members of lower central series with the help of commutator-associators of type (α, β) .

Theorem 5.4 from this paper is related to the main results of papers [16], [8]. But, unfortunately, these results from [16], [8] cannot be considered as proved. They are described in details in [15].

We will use the notation \square to mark the completion of a proof.

1 Preliminaries

Let us remind some notions and results from the loop theory, which can be found in [3] (see, also [1], [5]).

A *quasigroup* is a non-empty set Q together with a binary operation $Q \times Q \rightarrow Q; (x, y) \rightarrow xy$ such that the equations

$$ax = c, \quad yb = c \quad (1.1)$$

have unique solutions. A *loop* $(Q, \cdot, 1)$ is a quasigroup (Q, \cdot) with a base point, or distinguished element, $1 \in Q$ satisfying the equations $1x = x1 = x$ for all $x \in Q$.

The *multiplication group* $\mathfrak{C}(Q)$ of the arbitrary loop $Q = (Q, \cdot, 1)$ is generated by all mappings $R(x), L(x)$, where $R(x)y = yx, L(x)y = xy$, of the loop Q . The *inner mapping group* $\mathfrak{I}(Q)$ of Q is the subgroup of $\mathfrak{C}(Q)$, generated by the mappings $T(x), R(x, y), L(x, y)$, for all x, y in Q , where

$$\begin{aligned} T(x) &= L^{-1}(x)R(x), \quad R(x, y) = R^{-1}(xy)R(y)R(x), \\ L(x, y) &= L^{-1}(xy)L(x)L(y). \end{aligned} \quad (1.2).$$

For the loop Q with identity 1

$$\mathfrak{I}(Q) = \{\alpha \in \mathfrak{C}(Q) | \alpha 1 = 1\}. \quad (1.3)$$

The subloop H of the loop Q is called *normal* (*invariant*) in Q , if

$$xH = Hx, \quad x \cdot yH = xy \cdot H, \quad H \cdot xy = Hx \cdot y \quad (1.4)$$

or by (1.2)

$$T(x)H = H, \quad L(x, y)H = H, \quad R(x, y)H = H \quad (1.5)$$

for every $x, y \in Q$.

We will use the notation $\langle M \rangle$ for the subloop of the loop Q generated by set $M \subseteq Q$. Let H, K be a subloop of the loop Q such that K is normal

in $\langle H, K \rangle$. Then $\langle H, K \rangle = HK = KH$.

Lemma 1.1. *Let H be a normal subloop of the loop Q . If H is generated as a normal subloop by set $S \subseteq Q$, then H , as a subloop, is generated by set $\{\varphi s | s \in S, \varphi \in \mathfrak{I}(Q)\}$, where $\mathfrak{I}(Q)$ is the inner mapping group of Q .*

For arbitrary elements x, y, z of the loop Q the *commutator*, (x, y) , and *associator*, (x, y, z) , are defined by

$$xy = (yx)[x, y], \quad xy \cdot z = (x \cdot yz)[x, y, z]. \quad (1.6)$$

The *left nucleus*, $N_\lambda(Q)$, of the loop Q is the associative subloop $N_\lambda(Q) = \{a \in Q | [a, x, y] = 1 \quad \forall x, y \in Q\}$, the *middle nucleus*, $N_\mu(Q)$, of Q is the associative subloop $N_\mu(Q) = \{a \in Q | [x, a, y] = 1 \quad \forall x, y \in Q\}$ and the *right nucleus*, $N_\rho(Q)$, of Q is the associative subloop $N_\rho(Q) = \{a \in Q | [x, y, a] = 1 \quad \forall x, y \in Q\}$.

The *nucleus*, $N(Q)$, is defined by $N(Q) = N_\lambda(Q) \cap N_\mu(Q) \cap N_\rho(Q)$ and the *centre*, $Z(Q)$, of the loop Q is a normal associative and commutative subloop $Z(Q) = \{a \in N(Q) | [a, x] = 1 \quad \forall x \in Q\}$. Moreover, every subloop $H \subseteq Z(Q)$ is normal in Q .

The (transfinite) *upper central series*, $\{Z_\alpha\}$, of a loop Q is defined inductively as follows:

- i) $Z_0 = 1$;
- ii) for any ordinal α $Z_{\alpha+1}$ is the unique subloop of Q such that $Z_{\alpha+1}/Z_\alpha = Z(Q/Z_\alpha)$;
- iii) if α is a limit ordinal, Z_α is the union of all Z_β with $\beta < \alpha$.

For any normal subloop of a loop Q we define (N, Q) as the subloop A , whose existence is guaranteed by validity of condition: if A is the intersection of all normal subloops K of Q such that NK/K is a subloop of $Z(Q/K)$, then NA/A is a subloop of $Z(Q/A)$.

The (transfinite) *lower central series*, $\{Q_\alpha\}$, of the loop Q is defined inductively as follows:

- i) $Q_0 = Q$;
- ii) for any ordinal α $Q_{\alpha+1} = (Q_\alpha, Q)$;
- iii) if α is a limit ordinal, Q_α is the intersection of all Q_β with $\beta < \alpha$.

Clearly, Q_α, Z_α are normal subloops of Q .

The loop Q is called *transfinitely centrally nilpotent* (respect. *transfinitely upper centrally nilpotent*) if $Q_\lambda = 1$ (respect. $Z_\lambda = Q$) and *centrally nilpotent* (respect. *upper centrally nilpotent*) if, in addition, λ is finite. The

smallest ordinal λ such that $Q_\lambda = 1$, is called *centrally nilpotence class*.

The loop is *Moufang* if it satisfies the equivalent identities:

$$x(y \cdot xz) = (xy \cdot x)z; \quad (xy \cdot z)y = x(y \cdot zy); \quad xy \cdot zx = (xy \cdot z)x. \quad (1.7)$$

The Moufang loop is diassociative, i.e. every two elements generate a subgroup.

The Moufang loop is an *IP*-loop. Then it satisfies the identities:

$$x^{-1} \cdot xy = y; \quad yx \cdot x^{-1} = y; \quad (xy)^{-1} = y^{-1}x^{-1}; \quad (x^{-1})^{-1} = x. \quad (1.8)$$

Lemma 1.2. *In a Moufang loop G , the equation $[a, b, c] = 1$ implies each of the equations obtained by permuting a, b, c or replacing any of these elements by their inverses.*

Lemma 1.3. *Let a, b, c, d be four elements of the Moufang loop G each three of which associate (satisfy $[x, y, z] = 1$). Then the following equations are equivalent; (i) $[ab, c, d] = 1$; (ii) $[cd, a, b] = 1$; $[bc, d, a] = 1$.*

Lemma 1.4 *Let G be a Moufang loop. Then G satisfies all or none of the identities: (i) $[[x, y, z], x] = 1$; (ii) $[x, y, [y, z]] = 1$; (iii) $[x, y, z]^{-1} = [x^{-1}, y, z]$; (iv) $[x, y, z] = [x, z, y^{-1}]$. When these identities hold, the following identities hold:*

$$[x, y, z] = [y, z, x] = [y, x, z]^{-1}, \quad (1.9)$$

$$[xy, z] = [x, z][[x, z], y][y, z][x, y, z]^3, \quad (1.10)$$

In a Moufang loop the following identities hold:

$$L(z, y)x = x[x, y, z]^{-1}; \quad (1.11)$$

$$L(z, u)(xy) \cdot [z^{-1}, u^{-1}] = L(z, u)x \cdot (y \cdot L(z, u)[z^{-1}, u^{-1}]). \quad (1.12)$$

Let $(Q, \cdot, 1)$ be a *IP*-loop. Then $N(Q) = N_\lambda(Q) = N_\mu(Q) = N_\rho(Q)$ and $Z(Q) = \{a \in N(Q) \mid [a, x] = 1 \quad \forall x \in Q\}$.

R. H. Bruck and L. J. Paige [4] defined an *A-loop* to be a loop in which every inner mapping is an automorphism. Many of the basic theorems about *A*-loops are contained in [4]; for example, *A-loops are always power associative* (every $\langle x \rangle$ is a group), but not necessarily diassociative.

The following statements show an essential difference between *A*-loops and Moufang loops.

Proposition 1.5 [12]. *For an A-loop L , the following are equivalent:*

1. L is IP-loop;
2. L has the alternative property, i.e. $x(xy) = x^2y$ and $(yx)x = yx^2$ for all $x, y \in L$;
3. L is diassociative;
4. L is a Moufang loop.

Lemma 1.6 [3], [1]. For any IP-loop Q $N_\lambda(Q) = N_\rho(Q) = N_\mu(Q)$.

Lemma 1.7 [4]. For any A-loop Q $N_\lambda(Q) = N_\rho(Q) \subseteq N_\mu(Q)$.

2 Centrally nilpotent loops

Let Q be a loop. The series of normal subloops $Q = C_0 \supseteq C_1 \supseteq \dots \supseteq C_r = 1$ of loop Q is called *centrally nilpotent*, if

$$(C_i, Q) \subseteq C_{i+1} \quad \text{for all } i. \quad (2.1)$$

or, equivalently,

$$C_i/C_{i+1} \subseteq Z(Q/C_{i+1}) \quad \text{for all } i \quad (2.2)$$

Lemma 2.1. Let $\{C_i\}$ be a centrally nilpotent series, $\{Z_i\}$ be the upper centrally nilpotent series, $\{Q_i\}$ be the lower centrally nilpotent series of the loop Q . Then $C_{r-i} \subseteq Z_i$, $C_i \supseteq Q_i$, for $i = 0, 1, \dots, r$.

Proof. We have $C_0 = Q = Q_0$. Assume that $C_i \supseteq Q_i$. By (2.1) $(C_i, Q) \subseteq C_{i+1}$. But then $Q_{i+1} = (Q_i, Q) \subseteq (C_i, Q) \subseteq C_{i+1}$. We assume now that $C_{r-i} \subseteq Z_i$ for a certain i . Then the loop Q/Z_i is the homomorphic image of the loop Q/C_{r-i} with kernel Z_i/C_{r-i} . But by (2.2)

$$C_{r-i-1}/C_{r-i} \subseteq Z(Q/C_{r-i}),$$

from where it follows that the homomorphic image of subloop C_{r-i-1}/C_{r-i} must lie in the centre $Z(Q/Z_i)$. It is clear that this image is the subloop $(C_{r-i-1} \cup Z_i)/Z_i$, while $Z(Q/Z_i) = Z_{i+1}/Z_i$. Consequently, $C_{r-i-1} \subseteq C_{r-i} \cup Z_i \subseteq Z_{i-1}$. \square

Proposition 2.2. A loop Q is centrally nilpotent of class n if and only if its upper and lower centrally nilpotent series have respectively the form

$$1 = Z_0 \subset Z_1 \subset \dots \subset Z_n = Q, \quad Q = Q_0 \supset Q_1 \supset \dots \supset Q_n = 1.$$

Proof. The statement of the proposition for upper centrally nilpotent series results from the definition of centrally nilpotent loop. Further, if an centrally nilpotent series of length n exists, then from Lemma 2.1 it follows that the length of the upper and lower central series do not exceed n . But,

as there is a term by term inclusion between the elements of these series, their lengths are equal, and the series have the indicated form. \square

We follow [6]. Let Q be a loop and let $a, b, c \in Q$. We denote the solution of equation $ab \cdot c = ax \cdot bc$ (respect. $c \cdot ba = cb \cdot xa$) by $\alpha(a, b, c)$ (respect. $\beta(a, b, c)$) and call it the *associator of type α* (respect. *of type β*) of elements a, b, c .

The *commutator* of elements $a, b \in Q$ (a, b) is defined by the equality $ab = b(a(a, b))$.

The last definitions will be written in the form

$$T(b)a = a(a, b), R(b, c)a = a\alpha(a, b, c), L(c, b)a = \beta(a, b, c)a \quad (2.3)$$

if we use (1.2) (see, also, [3]).

Lemma 2.3 *Any IP-loop satisfies the identity*

$$\alpha(x, y, z)^{-1} = \beta(x^{-1}, y^{-1}, z^{-1}).$$

Proof. From the identity $xy \cdot z = x\alpha(x, y, z) \cdot yz$ by (1.8) we get $(xy \cdot z)^{-1} = (x\alpha(x, y, z) \cdot yz)^{-1}$, $z^{-1} \cdot y^{-1}x^{-1} = z^{-1}y^{-1} \cdot \alpha(x, y, z)^{-1}x^{-1}$ and from the definition of the associator of type β it follows that $\alpha(x, y, z)^{-1} = \beta(x^{-1}, y^{-1}, z^{-1})$. \square

Lemma 2.4. *Any Moufang loop Q satisfies the identities $[x, y, z]^{-1} = \alpha(x, z^{-1}, y^{-1})$, $[x, y, z] = \beta(x^{-1}, z, y)$, $[x, y] = (x, y)$.*

Proof. Let $a, b, c \in Q$. By (1.6) $ab \cdot c = (a \cdot bc)[a, b, c]$, $(ab \cdot c)[a, b, c]^{-1}$

$$= a \cdot bc, [a, b, c]^{-1} = (ab \cdot c)^{-1}(a \cdot bc),$$

$$[a, b, c]^{-1} = (c^{-1} \cdot b^{-1}a^{-1})(a \cdot bc)$$

and by the first identity from (1.7) we have

$$a[a, b, c]^{-1} = a((c^{-1} \cdot b^{-1}a^{-1})(a \cdot bc)) =$$

$$(a(c^{-1} \cdot b^{-1}a^{-1}) \cdot a)(bc) = ((ac^{-1})(b^{-1}a^{-1} \cdot a))(bc) = (ac^{-1} \cdot b^{-1})(bc).$$

Hence $a[a, b, c]^{-1} \cdot c^{-1}b^{-1} = ac^{-1} \cdot b^{-1}$. Further,

$$(a[a, b, c]^{-1} \cdot c^{-1}b^{-1})^{-1} = (ac^{-1} \cdot b^{-1})^{-1}, bc \cdot [a, b, c]a^{-1} = b \cdot ca^{-1}.$$

Then, from here and from the definition of the associator of type α or β , it follows that $[a, b, c]^{-1} = \alpha(a, c^{-1}, b^{-1})$, $[a, b, c] = \beta(a^{-1}, c, b)$.

Any Moufang loop is diassociative, so $[a, b] = (a, b)$. \square

Let Q be a loop and let A, B, C be non-empty subsets of Q . We denote $\alpha(A, B, C) = \langle \alpha(a, b, c) | a \in A, b \in B, c \in C \rangle$, $\beta(A, B, C) = \langle \beta(a, b, c) | a \in A, b \in B, c \in C \rangle$, $[A, B, C] = \langle [a, b, c] | a \in A, b \in B, c \in C \rangle$, $(A, B) = \langle (a, b) | a \in A, b \in B \rangle$, $[A, B] = \langle [a, b] | a \in A, b \in B \rangle$.

Lemma 2.5. *Let N be a normal subloop of the loop Q . Then the subloop H , generated by the set $\alpha(N, Q, Q) \cup \beta(N, Q, Q) \cup (N, Q)$ is normal in Q .*

Proof. Let $n \in N$, $x, y \in Q$. From (2.3) we get

$$\alpha(n, x, y) = L^{-1}(n)R(x, y)n,$$

$$\beta(n, x, y) = R^{-1}(n)L(x, y)n, (n, x) = L^{-1}(n)T(x)n. \quad (2.4)$$

The subloop N is normal in Q , then by (1.5) $\alpha(n, x, y), \beta(n, x, y), (n, x) \in N$ for all $n \in N$ and all $x, y \in Q$. Then $H \subseteq N$. Let $h \in H$. According to (2.3) we get $T(x)h = h(h, x) \in H$, $L(x, y)h = \beta(h, y, x) \in H$, $R(x, y)h = h\alpha(h, x, y) \in H$. Hence by (1.5) the subloop H is normal in Q . \square

Proposition 2.6. *A subloop H of a loop Q is normal in Q if and only if (i) $\alpha(H, Q, Q) \subseteq H$, $\beta(H, Q, Q) \subseteq H$, $(H, Q) \subseteq H$.*

Proof. Let H be a normal subloop of Q . Then from (1.5), (2.3) and Lemma 2.4 it follows that H satisfies the inclusions of Proposition 2.6. The inverse statement of Proposition 2.6 follows from Lemma 2.5. \square

Let H be a normal subloop of a loop Q . We denote $\mathcal{Z}_H(Q) = \{a \in Q | \alpha(a, Q, Q) \subseteq H, \beta(a, Q, Q) \subseteq H, (a, Q) \subseteq H\}$.

Let $E = \{1\}$ and let $\mathcal{Z}_E(Q) = \mathcal{Z}(Q)$. We prove that

$$\mathcal{Z}(Q) = Z(Q), \quad (2.5)$$

where $Z(Q)$ means the centre of loop Q . Really, from the definitions of the associators $\alpha(x, y, z)$, $\beta(x, y, z)$ and the nucleus of the loop it follows that $\alpha(a, x, y) = 1$ for all $x, y \in Q \Leftrightarrow a \in N_\lambda(Q)$, $\beta(a, x, y) = 1$ for all $x, y \in Q \Leftrightarrow a \in N_\rho(Q)$ and $(a, N_\lambda \cap N_\rho) = 1 \Rightarrow a \in N_\mu$. From here it follows that the equality (2.5) holds.

Particularly, if Q is a Moufang loop, then from Lemma 2.4 it follows that

$$a \in \mathcal{Z}(Q) \Leftrightarrow [a, x, y] = 1, [a, x] = 1 \quad \forall x, y \in Q. \quad (2.6)$$

If Q is an IP -loop or A -loop, then from Lemmas 1.6 and 1.7 it follows that

$$a \in \mathcal{Z}(Q) \Leftrightarrow \alpha(a, x, y) = 1, (a, x) = 1 \quad \forall x, y \in Q \quad (2.7)$$

or

$$a \in \mathcal{Z}(Q) \Leftrightarrow \beta(a, x, y) = 1, (a, x) = 1 \quad \forall x, y \in Q. \quad (2.8)$$

Lemma 2.7. *Let H be a normal subloop of the loop Q . Then the set $\mathcal{Z}_H(Q)$ is a normal subloop of the loop Q and $H \subseteq \mathcal{Z}_H(Q)$. Moreover, if N is a subloop of Q and $H \subseteq N \subseteq \mathcal{Z}_H(Q)$ then N is a normal subloop of Q .*

Proof. Let $\varphi : Q \rightarrow \overline{Q} = Q/H$ be the natural homomorphism. Denote by \overline{A} the image of set $A \subseteq Q$ under homomorphism φ . It is clear that $A \subseteq H$ if and only if $\overline{A} = \{1\}$. Particularly, the inclusion $\alpha(A, B, C) \subseteq H$ is equivalent to the equality $\alpha(\overline{A}, \overline{B}, \overline{C}) = \{1\}$. Hence, due to one-to-one correspondence between normal subloops of Q , containing H and all normal subloops of \overline{Q} to prove the Lemma 2.7 it is sufficient to consider that $H = E = \{1\}$. By (2.5) $\mathcal{Z}_E(Q)$ coincides with center $Z(Q)$, of loop Q , hence $\mathcal{Z}_E(Q)$ is a normal subloop of Q . Then due to our supposition $\mathcal{Z}_H(Q)$ is also a normal subloop of Q .

Now let N be a subloop of Q and $N \subseteq \mathcal{Z}_H(Q)$. Then $\alpha(N, Q, Q) \subseteq \alpha(\mathcal{Z}_E(Q), Q, Q) = E \subseteq N$. Similarly, $\beta(N, Q, Q) = N$, $(N, Q) = N$. By Lemma 2.5 N is a normal subloop of Q . \square

Let N be a normal subloop of the loop Q . We denote by $\mathcal{A}^N(Q)$ the subloop of Q generated by the set $\alpha(N, Q, Q) \cup \beta(N, Q, Q) \cup (N, Q)$. Particularly, if Q is a Moufang loop then $\mathcal{A}^N(Q)$ is generated by the set $[N, Q, Q] \cup [N, Q]$.

The subloop $\mathcal{A}^Q(Q)$ will be called *commutator-associator subloop* of the loop Q , and sometimes is denoted by $Q^{(1)}$. For an arbitrary loop Q the subloop $\mathcal{A}^Q(Q)$ is generated by the set $\alpha(Q, Q, Q) \cup \beta(Q, Q, Q) \cup (Q, Q)$. But if Q is a Moufang loop then $\mathcal{A}^Q(Q)$ is generated by the set $[Q, Q, Q] \cup [Q, Q]$ according to Lemma 2.4.

Lemma 2.8. *Let N be a normal subloop of the loop Q . Then the subloop $\mathcal{A}^N(Q)$ will be a normal subloop of the loop Q . Moreover, if H is a subloop of Q such that $N \supseteq H \supseteq \mathcal{A}^N(Q)$ then H will be a normal subloop of the loop Q .*

Proof. The subloop N is normal in Q , then by Proposition 2.6 $\alpha(N, Q, Q) \subseteq N$, $\beta(N, Q, Q) \subseteq N$, $(N, Q) \subseteq N$. Hence, $\mathcal{A}^N(Q) \subseteq N$. From here it follows that for a subloop H such that $N \supseteq H \supseteq \mathcal{A}^N(Q)$ we have $\alpha(H, Q, Q) \subseteq \alpha(N, Q, Q) \subseteq \mathcal{A}^N(Q) \subseteq H$. Analogically, $\beta(H, Q, Q) \subseteq H$, $(H, Q) \subseteq H$. Then by Proposition 2.6 the subloop H is normal in Q . \square

Corollary 2.9. *The commutator-associator subloop $\mathcal{A}^Q(Q)$ of a loop Q is the least normal subloop of Q such that the quotient loop $Q/\mathcal{A}^Q(Q)$ is an*

abelian group.

From the definitions of subloops \mathcal{Z}_N , \mathcal{A}^N and Lemmas 2.7, 2.8 it follows.

Corollary 2.10. *Let N be a normal subloop of the loop Q . Then the normal subloops $\mathcal{Z}_N(Q) = \mathcal{Z}_N$, $\mathcal{A}^N(Q) = \mathcal{A}^N$ satisfy the relations*

$$\mathcal{Z}_N/N = \mathcal{Z}(Q/N), \quad N/\mathcal{A}^N \subseteq \mathcal{Z}(Q/\mathcal{A}^N),$$

$$\mathcal{Z}_{\mathcal{A}^N} \supseteq N, \quad \mathcal{A}^{\mathcal{Z}_N} \subseteq N.$$

Lemma 2.11. *Let N be a normal subloop of the loop Q and let H be the normal subloop defined in Lemma 2.5. Then for any normal subloop K of Q $NK/K \subseteq Z(Q/K)$ if and only if $H \subseteq K$.*

Proof. As K is a normal subloop of the loop Q then by (1.4) $(xK)(yK \cdot nK) = (xK \cdot yK)(nK)$, $(nK \cdot xK)(yK) = (nK)(xK \cdot yK)$, $nK \cdot xK = xK \cdot nK$ for all $n \in N$ and all $x, y \in Q$ when and only when $(x \cdot yn)K = xy \cdot Kn$, $(nx \cdot y)K = nK \cdot xy$, $(nx)K = x \cdot nK$ respectively. But this is equivalent to $x \cdot yn \in xy \cdot Kn$, $nx \cdot y \in nK \cdot xy$, $nx \in x \cdot nK$ or $R^{-1}(n)L(x, y)n \in K$, $L^{-1}(n)R(x, y)n \in K$, $L^{-1}(n)T(x)n \in K$. The centre $Z(Q/K)$ of the loop Q/K is an abelian group, then by (2.4) $NK/K \subseteq Z(Q/K)$ when and only when $H \subseteq K$. \square

Proceeding to the description of upper, lower central series. From the definition of upper central series, (2.5) and Lemma 2.7 it follows that:

Proposition 2.12. *The transfinite upper central series $\{\mathcal{Z}_\alpha\}$ of a loop Q have the form:*

- i) $\mathcal{Z}_0 = 1$;
- ii) for any ordinal α $\mathcal{Z}_{\alpha+1}/\mathcal{Z}_\alpha = \mathcal{Z}_\alpha(Q/\mathcal{Z}_\alpha)$;
- iii) if α is a limit ordinal, $\mathcal{Z}_\alpha = \bigcup_{\beta < \alpha} \mathcal{Z}_\beta$.

Particularly, if Q is a commutative loop either an IP-loop, or a Moufang loop, or an A-loop then the normal subloops \mathcal{Z}_α change in accordance with the specifications shown after equalities (2.5) – (2.8).

From the definitions of transfinite lower central series, Lemmas 2.8, 2.11 and Corollary 2.9 it immediately follows that:

Proposition 2.13. *The transfinite lower central series $\{Q_\xi\}$ of a loop Q have the form:*

- $Q_0 = Q$, $Q_1 = \mathcal{A}^Q(Q)$, $Q_{\xi+1} = \mathcal{A}^{Q_\xi}(Q)$ for any ordinal ξ ;
- $Q_\xi = \bigcap_{\eta < \xi} Q_\eta$ if ξ is a limit ordinal.

If Q is a Moufang loop then the normal subloops change in accordance with the definition of the subloop $\mathcal{A}^N(Q)$, where N is a normal subloop of

Q .

Now we define the *commutator-associator of type (α, β) of weight n* inductively:

1) any associators of the form $\alpha(x, y, z)$, $\beta(x, y, z)$ and any commutator (x, y) are commutator-associator of the type (α, β) of weight 1;

2) if a is a commutator-associator of the type (α, β) of weight $n - 1$, then $\alpha(a, x, y)$, $\beta(a, x, y)$, (a, x) , where $x, y \in Q$, are a commutator-associator of the type (α, β) of the weight n .

If only the associators of types α and β participate in the definition, then we get the *associators of type (α, β)* .

We define by induction the *commutator-associator of type (μ) of weight n* :

1) any associator of form $[x, y, z]$ and any commutator $[x, y]$ are commutator-associators of type (μ) of weight 1;

2) if a is a commutator-associator of type (μ) of weight $n - 1$, then $[a, x, y]$, $[a, x]$ are commutator-associators of type (μ) of weight n .

If only the associators of type μ participate in the definition then we get the *associators of type (μ)* .

We assume that in the loop Q all commutator-associators of weight n of one of the aforementioned types, for example (α, β) , are equal to unit 1. Then we say that the loop Q satisfies the *commutator-associator identities* of type (α, β) of weight n . The notions of *associator identities* of type (α, β) of weight n , of *associator identities* of type (α) of weight n , *associator identities* of type (β) of weight n , of *associator identities* of type (μ) of weight n are obvious.

Further we denote by $W_n(\alpha, \beta)$ the set of all commutator-associators of type (α, β) of weight n , by $W_n(\alpha)$ the set of all commutator-associators of type (α) of weight n , by $W_n(\beta)$ the set of all commutator-associators of type (β) of weight n , by $W_n(\mu)$ the set of all commutator-associators of type (μ) of weight n , and so on. Reinforce the Proposition 2.2 and Proposition 2.4 in case of centrally nilpotent loops using the Propositions 2.12, 2.13.

Theorem 2.14. *For a loop Q the following statements are equivalent:*

- 1) *the loop Q is centrally nilpotent of class n ;*
- 2) *the upper central nilpotent series of Q have the form*

$$E = \mathcal{Z}_0 \subset \mathcal{Z}_1 \subset \mathcal{Z}_2 \subset \dots \subset \mathcal{Z}_{n-1} \subset \mathcal{Z}_n = Q,$$

where $E = \{1\}$, $\mathcal{Z}_{i+1}/\mathcal{Z}_i = \mathcal{Z}(Q/\mathcal{Z}_i)$, $i = 0, \dots, n - 1$, and

$$\mathcal{Z}_i = \{a \in Q \mid w(a, x_1, \dots, x_j) = 1 \forall w \in W_i(\alpha, \beta), \forall x_1, \dots, x_j \in Q\}. \quad (2.9)$$

3) the lower central nilpotent series of Q have the form

$$Q = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \dots \supset \mathcal{A}_{n-1} \supset \mathcal{A}_n = \{1\},$$

where $\mathcal{A}_{i+1} = \mathcal{A}^{\mathcal{A}_i}(Q)$ for $i = 0, 1, \dots, n-1$ and the normal subloop \mathcal{A}_i coincides with the subloop A_i generated by all commutator-associators $w \in W(\alpha, \beta)$.

Particularly, it is sufficient to consider $w \in W_i(\mu)$ if Q is a Moufang loop and $w \in W_i(\alpha)$ or $w \in W_i(\beta)$ if Q is an IP-loop or A-loop.

Proof. The equivalence of items 1) - 3) follows from Proposition 2.2.

By definition $\mathcal{Z}_1 = \{a \in Q \mid \alpha(a, x, y) = 1, \beta(a, x, y) = 1, (a, x) = 1 \forall x, y \in Q\}$. From $\mathcal{Z}_{i+1}/\mathcal{Z}_i = \mathcal{Z}(Q/\mathcal{Z}_i)$ it follows that $\mathcal{Z}_{i+1}/\mathcal{Z}_i = \{a \in Q \mid \alpha(a, x, y) \subseteq \mathcal{Z}_i, \beta(a, x, y) \subseteq \mathcal{Z}_i, (a, x) \subseteq \mathcal{Z}_i \forall x, y \in Q\}$. From here by induction it follows easily that (2.6).

From (2.9) it follows that all commutator-associators of type (α, β) of weight $n-1$ belong to the normal subloop \mathcal{Z}_1 , which according to (2.6) and Lemmas 1.6, 1.7 coincides with centre $Z(Q)$ of loop Q . Then the subloop \mathcal{A}_{n-1} , generated by these commutator-associators, is normal in Q .

We assume that the subloop \mathcal{A}_{i+1} is normal in Q and we consider the natural homomorphism $\varphi : Q \rightarrow Q/\mathcal{A}_{i+1}$. The subloop $\mathcal{A}_i/\mathcal{A}_{i+1}$ belongs to the centre $Z(Q/\mathcal{A}_{i+1})$, then it is normal in Q/\mathcal{A}_{i+1} . The inverse image of $\mathcal{A}_i/\mathcal{A}_{i+1}$ under homomorphism φ is \mathcal{A}_i . Hence, the subloop \mathcal{A}_i is normal in Q . \square

Corollary 2.15. *According to Proposition 2.13 let $\{\mathcal{A}_\xi\}$, where $\mathcal{A}_{\xi+1} = \mathcal{A}^{\mathcal{A}_\xi}(Q)$, be the (transfinite) lower central series of a loop Q . Then for any natural number n the normal subloop \mathcal{A}_n of the series $\{\mathcal{A}_\xi\}$ coincides with the subloop A_n of the loop Q , generated by all commutator-associators of type (α, β) of weight n and the quotient loop Q/\mathcal{A}_n is centrally nilpotent of class $\leq n$.*

Proof. From Lemma 1.1, (1.4), (2.3) it follows that the subloop \mathcal{A}_n is normal in Q .

By definition, $\mathcal{A}_1 = \mathcal{A}^Q(Q)$ and from Lemma 2.8 it follows that \mathcal{A}_1 is the subloop of Q generated by all commutator-associators of type (α, β) of weight 1. Hence \mathcal{A}_1 is the normal subloop of Q generated by all commutator-associators of type (α, β) of weight 1, i.e. $\mathcal{A}_1 = A_1$.

We consider the normal subloops \mathcal{A}_{n+1} and \mathcal{A}_{n+1} . By construction, \mathcal{A}_{n+1} is the subloop of Q generated by the set $\alpha(\mathcal{A}_n, Q, Q) \cup \beta(\mathcal{A}_n, Q, Q) \cup (\mathcal{A}_n, Q)$. By inductive hypothesis $\mathcal{A}_n = A_n$. Then the set $\alpha(\mathcal{A}_n, Q, Q) \cup \beta(\mathcal{A}_n, Q, Q) \cup (\mathcal{A}_n, Q)$ contain all commutator-associators of type $(\alpha, \beta, 1)$

of weight $n + 1$. Hence $\mathcal{A}_{n+1} \supseteq A_{n+1}$ because \mathcal{A}_{n+1} is a normal subloop in Q .

Taking this into consideration, we consider the quotient loop Q/A_{n+1} . In this loop all commutator-associators of type (α, β) of weight $n + 1$ are equal to unit. Then by (2.5) all commutator-associators of weight n will be in the centre of the loop Q/A_{n+1} . Consequently, $\mathcal{A}_n/A_{n+1} = A_n/A_{n+1} \subseteq \mathcal{Z}(Q/A_{n+1})$ as the centre of any loop is a normal subloop. Proceeding to inverse images we get $\alpha(\mathcal{A}_n, Q, Q), \beta(\mathcal{A}_n, Q, Q), (\mathcal{A}_n, Q) \subseteq A_{n+1}$. Hence $\mathcal{A}_{n+1} \subseteq A_{n+1}$. Consequently, $A_{n+1} = \mathcal{A}_{n+1}$.

From $\mathcal{A}_n = A_n$ it follows that any commutator-associators of type (α, β) of weight n of Q/\mathcal{A}_n is equal to unit. Then by Theorem 2.14 the loop Q/\mathcal{A}_n is centrally nilpotent of class $\leq n$. \square

Reinforce the Corollary 2.15 in case of Moufang loops and A-loops.

Proposition 2.16. *Let Q be a Moufang loop (respect. IP-loop or A-loop) and according to Proposition 2.13 let $\{\mathcal{A}_\xi\}$, where $\mathcal{A}_{\xi+1} = \mathcal{A}^{\mathcal{A}_\xi}(Q)$, be the (transfinite) lower central series of a loop Q . Then for any natural number n the normal subloop \mathcal{A}_n of the series $\{\mathcal{A}_\xi\}$ coincides with the subloop A_n of the loop Q , generated by all commutator-associators of type (μ) (respect. of type α or β) of weight n and the quotient loop Q/\mathcal{A}_n is centrally nilpotent of class $\leq n$.*

Proof. It is easy to show by induction on natural number n the equality $\mathcal{A}_n = A_n$ using the Lemmas 2.3 and 2.4. Then the Proposition 2.16 follows from Corollary 2.15. \square

Corollary 2.17. *A loop Q is centrally nilpotent of class n when and only when it satisfies all commutator-associators identities $w(x_1, \dots, x_j) = 1$ of type (α, β) of weight n , $w \in W_n(\alpha, \beta)$, but does not satisfy at least one identity $v(x_1, \dots, x_l) = 1$ of type (α, β) of weight $n - 1$, $v \in W_{n-1}(\alpha, \beta)$.*

Particularly, if Q is a Moufang loop, then it is sufficient to consider $w \in W_n(\mu)$, $v \in W_{n-1}(\mu)$ and if Q is an IP-loop or A-loop, then it is sufficient to consider $w \in W_n(\alpha)$, $v \in W_{n-1}(\alpha)$ or $w \in W_n(\beta)$, $v \in W_{n-1}(\beta)$.

The statements that follow from the equivalence of items 1), 3) of Theorem 2.14, are true.

3 On varieties of Moufang loops and A-loops

The notion of loop $(Q, \cdot, 1)$ defined by one basic operation (see (1.1)) is not algebraical. In order to apply the universal algebraic techniques, one must

use the universal algebraic description of loops as algebras $(Q, \cdot, /, \backslash, 1)$ with three binary operations, multiplication \cdot , left division \backslash , right division $/$ and one nullary operation 1 , satisfying the identities

$$(x \cdot y)/y = x; (x/y) \cdot y = x, \quad x \backslash (x \cdot y) = y, \quad x \cdot (x \backslash y) = y, \quad (3.1)$$

$$1x = x1 = x. \quad (3.2)$$

Proposition 3.1. *Every quasigroup (respect. loop) $(Q, \cdot, /, \backslash)$ defined by three basic binary operations is a quasigroup (respect. loop) (Q, \cdot) defined by one basic binary operation.*

Proof. For quasigroup $(Q, \cdot, /, \backslash)$ we consider the equation $ax = b$, where $a, b \in Q$. Then by (3.1) $a \backslash (ax) = a \backslash b$, $x = a \backslash b$. If $ab = ac$, where $a, b, c \in Q$, then $a \backslash (ab) = a \backslash (ac)$, $b = c$. Hence the equation $ax = b$ has an unique solution. Similarly it is proved that the equation $xa = b$ has an unique solution. Hence (Q, \cdot) is a quasigroup with one basic operation. This completes the proof of Proposition 3.1. \square

Let $(Q, \cdot, 1)$ be a Moufang loop. According to the definition it satisfies the equivalence identities (1.7) and the identities (1.8). We denote $x \backslash y = x^{-1}y$, $x/y = xy^{-1}$. Then the algebra $(Q, \cdot, /, \backslash, 1)$ satisfies the identities (3.1) and (1.5). Hence $(Q, \cdot, /, \backslash, 1)$ is a Moufang loop. From here and Proposition 3.1 it follows that for Moufang loops the definition of loop with three basic binary operations and the definition of loop with one binary operation coincide.

Proposition 3.2. *The set \mathfrak{M} of all Moufang loops defined by three basic binary operations form a variety defined by one from equivalent identities (1.5).*

Obviously, from (3.2) it follows $x/x = x \backslash x = 1$, $1 \backslash x = x/1 = x$ for every x in Q , here 1 is an identity for (Q, \cdot) , a left identity for (Q, \backslash) , a right identity for $(Q, /)$. Assume that 1 is a left identity for $(Q, /)$, $1/x = x$. Then by (3.1) $(1/x)x = xx$, $1 = xx = x^2$. \square

We analyze the concept of A -loop. If for loop $(Q, \cdot, /, \backslash, 1)$ we consider the group generalized by all left and right translations with respect to basic operations $\cdot, /, \backslash$ and enter the concept of group of inner mappings, then by above-stated we shall receive the identity $x^2 = 1$. The A -loops with identity $x^2 = 1$ are investigated in [10]. We exclude this case. Moreover, in order not to limit the Bruck and Paige's concept of A -loop $(Q, \cdot, 1)$ we define.

Let $(Q, \cdot, /, \backslash, 1)$ be a loop, let $\mathfrak{C}(Q)$ be the multiplication group of loop $(Q, \cdot, 1)$ and let $\mathfrak{I}(Q)$ be the group of inner mapping of loop $(Q, \cdot, 1)$. The

loop $(Q, \cdot, /, \backslash, 1)$ will called *A-loop* if the inner mappings in $\mathfrak{I}(Q) \subseteq \mathfrak{E}(Q)$ are automorphism for loop $(Q, \cdot, 1)$.

Every *A-loop* is power associative [4], then $1/x = x \backslash 1$. Further for *A-loops* we shall use the designation

$$x^{-1} = 1/x = x \backslash 1. \quad (3.3)$$

For a loop $(Q, \cdot, /, \backslash, 1)$ we have $L(x)y = R(y)x = xy$. Denote $L_{(/)}(x)y = R_{(/)}(y)x = x/y$, $L_{(\backslash)}(x)y = R_{(\backslash)}(y)x = x \backslash y$. From (3.1) it follows that $x/(y \backslash x) = y$, $(x/y) \backslash x = y$, $L_{(/)}(x)R_{(\backslash)}(x)y = y$, $R_{(\backslash)}(x)L_{(/)}(x)y = y$. Similarly, from (3.1) it follows that $R_{(/)}R(y)(x) = x$, $L_{(\backslash)}L(x)y = y$. Hence

$$L^{-1}(x) = L_{(\backslash)}(x), \quad R^{-1}(x) = R_{(/)}(x), \quad L_{(/)}(x) = R_{(\backslash)}^{-1}(x). \quad (3.4)$$

The group of inner mapping $\mathfrak{I}(Q)$ of loop $(Q, \cdot, /, \backslash, 1)$ is generated by inner mapping $T(x) = L^{-1}(x)R(x)$, $R(x, y) = R^{-1}(xy)R(y)R(x)$, $L(x, y) = L^{-1}(xy)L(x)L(y)$ by (1.1) or according to (3.4) is generated by inner mapping

$$\begin{aligned} \overline{T}(x) &= L_{(\backslash)}(x)R(x), \quad \overline{R}(x, y) = R_{(/)}(xy)R(y)R(x), \\ \overline{L}(x, y) &= L_{(\backslash)}(xy)L(x)L(y). \end{aligned} \quad (3.5)$$

From (3.4), (3.5) follows that

$$\begin{aligned} \overline{T}^{-1}(x) &= R_{(/)}(x)L(x), \quad \overline{R}^{-1}(x, y) = R_{(\backslash)}(x)R_{(\backslash)}(y)R(xy), \\ \overline{L}^{-1}(x, y) &= L_{(\backslash)}(y)L_{(\backslash)}(x)L(xy). \end{aligned} \quad (3.6)$$

The group of inner mapping $\mathfrak{I}(Q)$ of loop $(Q, \cdot, /, \backslash, 1)$ is generated by inner mapping $T(x)$, $R(x, y)$, $L(x, y)$. Then from (3.5), (3.6) it follows that any inner mapping $\alpha \in \mathfrak{I}(Q)$ has a representation $\alpha = S_1 S_2 \dots S_n$, where

$$S_i \in \{\overline{T}(x_1), \overline{R}(x_2, x_3), \overline{L}(x_4, x_5), \overline{T}^{-1}(x_6), \overline{R}^{-1}(x_7, x_8), \overline{L}^{-1}(x_9, x_{10})\},$$

$x_1, \dots, x_{10} \in Q$. We choose the least number n with such a property and we fix such a record. The number $n = n(\alpha)$ will called a *length* of mapping α .

Let $(Q, \cdot, /, \backslash, 1)$ be an *A-loop*. Then $\alpha(u \cdot u) = \alpha(u) \cdot \alpha(v)$ for all $u, v \in Q$. We transform the last expression into loop expression $\overline{\alpha}(u \cdot u) = \overline{\alpha}(u) \cdot \overline{\alpha}(v)$ with respect to basic operations $\cdot, /, \backslash$ with the help of relations $L_{(\backslash)}(x)y = x \backslash y$, $R_{(/)}(x)y = y/x$. Further, $\overline{\alpha}(u \cdot u) = \overline{\alpha}(u) \cdot \overline{\alpha}(v)$ turns to identity if to

consider $\cdot, /, \backslash$ as symbols of operations and x_1, \dots, x_{10}, u, v as free variables (see [14, pag. 268]). Obviously that received identity of length n is valid in loop $(Q, \cdot, /, \backslash, 1)$.

We denote by \mathfrak{A}_n the variety of all A -loops $(Q, \cdot, /, \backslash, 1)$ defined by above described identities of length $\leq n$. We get $\mathfrak{A}_1 \supseteq \mathfrak{A}_2 \supseteq \dots \mathfrak{A}_n \supseteq \dots$. Obviously, $\mathfrak{A} = \cap \mathfrak{A}_n$ is a variety and is the variety of all A -loops with three basic operations. Is easy to show induction on n that every identity $\overline{\alpha}(u \cdot u) = \overline{\alpha}(u) \cdot \overline{\alpha}(v)$ of length n is a consequence of identity of length 1. Then $\mathfrak{A} = \mathfrak{A}_n$. Hence, according to (3.5), (3.6), we proved.

Proposition 3.3. *The set \mathfrak{A} of all A -loops defined by three basic binary operations form a variety defined by six identities $S_i(u \cdot v) = S_i u \cdot S_i v$, where $S_i \in \{\overline{T}(x_1), \overline{R}(x_2, x_3), \overline{L}(x_4, x_5), \overline{T}^{-1}(x_6), \overline{R}^{-1}(x_7, x_8), \overline{L}^{-1}(x_9, x_{10})\}$.*

Now we follow Mal'cev [14].

1). Any variety of algebras \mathfrak{B} is quite determined by its free algebra of countable infinite rank $F = f(\mathfrak{B})$ (Theorem VI.13.3). Let \mathfrak{C} be a subvariety of \mathfrak{B} . The least congruence θ on F such that $F/\theta \in \mathfrak{C}$ is called *word congruence*.

2). The quotient algebra F/θ is a free algebra of \mathfrak{C} .

3). A congruence θ on F is word congruence if $x\theta y$ implies $\varphi x \theta \varphi y$ for all endomorphisms $\varphi; F \rightarrow F$. The congruence θ with such an implication is called *fully characteristic* (Corollary VI.14.3).

4). Let $KI(F/\theta)$ denote the variety generated by F/θ . Then the mapping $\theta \rightarrow KI(F/\theta)$ is an antiisomorphism of lattice of fully characteristic congruences and lattice of all subvarieties of \mathfrak{B} (Corollary VI.14.4).

Further we shall transfer the above-stated statements on a variety \mathfrak{L} of all loops defined by three basic binary operations \mathfrak{L} and \mathfrak{L} -free loop $F = F(\mathfrak{L})$ with infinite set of free generators f_1, f_2, \dots .

5). According to [14, pag. 61] a relation of equivalence θ of algebra (A, Ω) is a congruence if θ is stable concerning all basic operations in Ω .

6). If $(A, \Omega) \in \mathfrak{L}$ then the conditions that θ is stable concerning all operations $(\cdot), (/), (\backslash): x_1 \theta x_2, y_1 \theta y_2 \rightarrow x_1 \alpha y_1 \theta x_2 \alpha y_2$, where $\alpha \in \{(\cdot), (/), (\backslash)\}$, is equivalent to condition: $x \theta y \Leftrightarrow xz \theta yz, zx \theta zy$ for all $z \in Q$.

7). The relation θ with the least condition for loops $(Q, \cdot, 1)$ in the literature [3], ([1]) is called *normal congruence* and the inverse image H of unity at homomorphism $Q \rightarrow Q/\theta$ is called *normal subloop* of loop $(Q, \cdot, 1)$. It coincides with the notion of normal subloop defined by (1.2).

8). From item 7) it follows that all researches of previous sections for

loops defined by one basic binary operation literally are valid for loops defined by three basic binary operations.

9). According to items 3) and 7) we define. Let \mathfrak{B} be a variety of loops defined by three basic binary operations and let $L = L(\mathfrak{B})$ be a \mathfrak{B} -free loop with infinite number of free generators. A normal subloop H of loop L will be called *fully invariant* if $\varphi H \subseteq H$ for all endomorphism $\varphi : L \rightarrow L$.

10). Let X_∞ denote the free loop with infinite set of free generators x_1, x_2, \dots of variety of all loops defined by three basic binary operations. This free loop we shall use as totality of "words": *word* means an element in X_∞ in the alphabet x_1, x_2, \dots .

If Q is a loop and α is a mapping of free generators x_1, x_2, \dots into Q then the image of word $v = v(x_{i_1}, \dots, x_{i_k}) \in X_\infty$ at homomorphism $\alpha : X_\infty \rightarrow Q$ is called *the value* of word v in Q , $v(\alpha x_{i_1}, \dots, \alpha x_{i_k})$.

The word $v(x_{i_1}, \dots, x_{i_k})$ is called an identity for Q if the unity 1 is the unique value of v in Q , $v(a) = 1$ for all $a \in Q^k$.

11). Let $(Q, \cdot, /, \backslash, 1)$ be a loop, let φ be an endomorphism of $(Q, \cdot, /, \backslash, 1)$, let $\mathfrak{I}(Q)$ be the group of inner mappings of loop $(Q, \cdot, 1)$ and let $\alpha \in \mathfrak{I}(Q)$. Then $\varphi \mathfrak{I}(Q)x \subseteq \mathfrak{I}(Q)\varphi x$ for all $x \in Q$.

Indeed, the inner mapping $\alpha \in \mathfrak{I}(Q)$ is a product of finite number of factors $T^{\pm 1}(a)$, $L^{\pm 1}(a, b)$, $R^{\pm 1}(a, b)$, $a, b \in Q$. Then the assertion follows from the definition of endomorphism φ of $(Q, \cdot, /, \backslash, 1)$ and (3.4) – (3.6).

12). Let \mathfrak{U} denote the variety of all loops defined by three basic binary operations with \mathfrak{U} -free loop $F = f(\mathfrak{U})$ of infinite number of free generators f_1, f_2, \dots and let \mathfrak{B} be a subvariety of \mathfrak{U} with \mathfrak{B} -free loop $G = G(\mathfrak{U})$ of infinite number of free generators g_1, g_2, \dots . A normal subloop H of loop $Q \in \mathfrak{B}$ will be called a *word subloop* if H is the normal subloop of Q generated as normal subloop by all values of words of some given set M : $M(Q)$ is the loop generated as normal subloop of Q by set $\{\varphi v | v \in M, \varphi \in \text{hom}(X_\infty, Q)\}$.

13). All words from M are identities for a loop $Q \in \mathfrak{B}$ if and only if $M(Q) = \{1\}$. If $v \in M$ and $v = v(x_1, \dots, x_k)$, then $M(Q) = \{v(a_1, \dots, a_k) | \forall a_1, \dots, a_k \in Q\}$.

14). Let M be some set of words. The word u is said to be a consequence of M in a variety \mathfrak{B} if with accuracy up to renameng of variables the word subloop generated by word u is a subloop of word subloop generated by set of word M in a \mathfrak{B} -free loop. Two sets of words M_1 and M_2 are equivalent in variety \mathfrak{B} if every word of M_1 is a consequence of M_2 and conversely.

15). A normal subloop H of the loop $Q \in \mathfrak{B}$ defined in item 12) is a

word subloop of Q if and only if H is fully characteristic in Q .

The proof of this statement is the items 16) – 18).

16). Every word subloop H of loop $Q \in \mathfrak{B}$ is fully invariant in Q .

The assertion follows from the definition of word subloop, Lemma 1.3 and item 11).

17). Every fully invariant subloop V of free loop $F = F(\mathfrak{U})$ defined in item 12) is word subloop in F .

Indeed, the subloop V is fully invariant in F . In such a case it is possible to consider as set of words M the set of such words $v = v(x_{i_1}, \dots, x_{i_k}) \in X_\infty$ that $v = v(f_{i_1}, \dots, f_{i_k}) \in V$.

18). Every fully invariant subloop of free loop $G = G(\mathfrak{B})$ defined in item 12) is word subloop in G .

Indeed, as $\mathfrak{B} \subseteq \mathfrak{U}$ then any mapping φ of free generators f_1, f_2, \dots on free generators g_1, g_2, \dots proceeds up to epimorphism $\varphi : F \rightarrow G$ which results in representation $G \cong F/R$ with property: $r(f) \in R$ for some $f = (f_{i_1}, \dots, f_{i_k})$ implies $r(\theta f) \in R$ at any endomorphism θ of loop F . Hence G can be presented as quotient loop $G \cong F/R$ on some of its fully characteristic subloop.

Let N be the inverse image in F of given fully invariant subloop of G . Then the subloop N/R is fully invariant in F/R . Every endomorphism of loop F induces an endomorphism of loop G , hence $\theta R \subseteq R$, $\theta N/R \subseteq N/R$. From the definition of induced endomorphism it follows that $\theta N \subseteq N$. Hence, the subloop N is fully invariant in F and by item 17) a verbal subloop in F : $N = M(F)$ for some set M of words. But then $N/R = M(F/R)$, as required. \square

According to Propositions 3.2, 3.3 and item 8) we consider the variety \mathfrak{U} of all loops either the variety \mathfrak{M} of all Moufang loops, or the variety \mathfrak{A} of all A -loops defined by three basic binary operations. Let F denote the \mathfrak{U} -free either \mathfrak{M} -free, or \mathfrak{A} -free loop on a countable infinite set of generators and let $\{F_\xi\}$ be the (transfinite) lower central series of F . From Corollaries 2.15 and 2.17, Proposition 32.16 and above-stated items 1) – 18) it follows.

Theorem 3.4. *Let n be a finite natural number and let F_n be the member of lower central series $\{F_\xi\}$ of free loop F . Then*

1) F_n is a fully invariant subloop of loop F generated by all commutator-associators of type (α, β) of weight n .

2) If \mathfrak{N} denotes the variety of all centrally nilpotent loops of class $\leq n$ in \mathfrak{U} , then the quotient loop F/F_n is a free loop of variety $\mathfrak{N} \subseteq \mathfrak{U}$.

3) Let $W_n(\alpha, \beta)$ (respect. $W_n(\mu)$) denote the set of all commutator-associators of type (α, β) of weight n (respect. the set of all commutator-associators of type (μ) of weight n either the set of all commutator-associators of type (α) of weight n or the set of all commutator-associators of type (β) of weight n) of loop X_∞ . Then the variety $\mathfrak{N} \supseteq \mathfrak{U}$ is determined by the finite set of identities $w = 1$, where $w \in W_n(\alpha, \beta)$. Particularly, if \mathfrak{U} is the variety of all Moufang loops, then $w \in W_n(\mu)$ and if \mathfrak{U} is the variety of all A -loops or all IP-loops, then $w \in W_n(\alpha)$ or $w \in W_n(\beta)$.

4 Lower central series of Moufang loops and A -loops

According to Proposition 2.13 and item 8) of Section 3 let

$$Q \supseteq \mathcal{A}_1(Q) \supseteq \dots \supseteq \mathcal{A}_k(Q) \supseteq \mathcal{A}_{k+1}(Q) \supseteq \mathcal{A}_{k+2}(Q) \supseteq \dots, \quad (4.1)$$

where $\mathcal{A}_{\xi+1}(Q) = \mathcal{A}^{\mathcal{A}_\xi}(Q)$, be the (transfinite) lower central series of a loop $(Q, \cdot, /, \backslash, 1)$. Let k be a finite natural number and let $\mathcal{A}_i(Q) = \mathcal{A}_i$. By Corollary 2.10 and (2.5)

$$\mathcal{A}_{k+1}/\mathcal{A}_{k+2} \subseteq \mathcal{Z}(Q/\mathcal{A}_{k+2}) = Z(Q/\mathcal{A}_{k+2}). \quad (4.2)$$

We consider the quotient loop $Q/\mathcal{A}_{k+2} = G$ and we suppose that the image of series (4.1) under homomorphism $\varphi : Q \rightarrow Q/\mathcal{A}_{k+2}$ has the form

$$Q/\mathcal{A}_{k+2} \supset \mathcal{A}_1/\mathcal{A}_{k+2} \supset \mathcal{A}_{k+1}/\mathcal{A}_{k+2} \supset \mathcal{A}_{k+2}/\mathcal{A}_{k+2} = 1.$$

Let $a \in \mathcal{A}_{k+2}$, $x, y \in Q$. Then by Corollary 2.15 $\alpha(a, x, y), \beta(a, x, y), (a, x) \in \mathcal{A}_{k+1}$ and by (1.4) $\alpha(a, x, y)\mathcal{A}_{k+2}, \beta(a, x, y)\mathcal{A}_{k+2}, (a, x)\mathcal{A}_{k+2} \in \mathcal{A}_{k+1}/\mathcal{A}_{k+2}$. From relation (4.2) $\mathcal{A}_{k+1}/\mathcal{A}_{k+2} \subseteq \mathcal{Z}(Q/\mathcal{A}_{k+2})$ we get

$$\alpha(a, x, y)\mathcal{A}_{k+2}, \beta(a, x, y)\mathcal{A}_{k+2}, (a, x)\mathcal{A}_{k+2} \in \mathcal{Z}(Q/\mathcal{A}_{k+2}). \quad (4.3)$$

According to Corollary 2.15 let $G \supset \mathcal{A}_1(G) \supset \mathcal{A}_2(G) = e$ be the lower central series of loop G . If H is a normal subloop of a loop L then by (1.4) $\alpha(xH, yH, zH) = \alpha(x, y, z)H$, $\beta(xH, yH, zH) = \beta(x, y, z)H$, $(xH, yH) = (x, y)H$. By Corollary 2.15 $\mathcal{A}_t(L) = \langle w_t(x_1, \dots, x_j) \mid \forall w_t \in W_t(\alpha, \beta) \forall x_1, \dots, x_j \in L \rangle$, $\mathcal{Z}_t(L) = \{a \in L \mid w_t(a, x_1, \dots, x_r) = 1 \quad \forall w_t \in W_t(\alpha, \beta) \forall x_1, \dots, x_r \in L\}$. From here it follows that

$$\mathcal{A}_{k+2}(Q) = \varphi^{-1}(e), \mathcal{A}_{k+1}(Q) \subseteq \varphi^{-1}(\mathcal{A}_1(G)), \mathcal{A}_k(Q) \subseteq G. \quad (4.4)$$

From (4.2), (4.3) it follows.

Lemma 4.1. *Let Q be a loop with (transfinite) lower central series $\{\mathcal{A}_\xi(Q)\}$ and let k be a natural number. Then $a \in \mathcal{A}_k(Q) \Rightarrow \alpha(a, x, y)\mathcal{A}_{k+2}(Q), \beta(a, x, y)\mathcal{A}_{k+2}(Q), (a, x)\mathcal{A}_{k+2}(Q) \in Z(Q/\mathcal{A}_{k+2})(Q), \forall x, y \in Q$.*

Lemma 4.2. *Let G be a centrally nilpotent loop of class 2. Then $\alpha(a, x, y), \beta(a, x, y), (a, x) \in Z(G) \forall a, x, y \in G$, where $Z(G)$ means the centre of loop G .*

The Lemma 4.2 follows from Lemma 4.1. It also easy follows from Corollary 2.17 and (2.9).

From (4.2) it follows that the quotient loop $\mathcal{A}_{k+1}/\mathcal{A}_{k+2}$ is an abelian group. Then for $u, v \in \mathcal{A}_{k+1}$ we will write the expressions $(u \setminus v) \pmod{\mathcal{A}_{k+2}}, (u/v) \pmod{\mathcal{A}_{k+2}}$ in the form $(u^{-1}v) \pmod{\mathcal{A}_{k+2}}, (uv^{-1}) \pmod{\mathcal{A}_{k+2}}$ respectively.

The crucial result used in the proof of Lemma 4.3 is the Lemma 4.2. We will use it without mentioning it and for Moufang loops we will use Lemma 4.2.

Lemma 4.3. *Let G be a centrally nilpotent loop of class 2 and let $a, b, x, y, z \in G$. If G is a Moufang loop then*

$$[a, x, y] = [x, y, a] = [y, a, x], \quad (4.5)$$

$$[a, x, y]^{-1} = [a^{-1}, x, y] = [a, x^{-1}, y] = [a, y, x], \quad (4.6)$$

$$[ab, x] = [a, x][b, x][a, b, x]^3, [a, xy] = [a, x][a, y][a, x, y]^3, \quad (4.7)$$

$$[a, xy, z] = [a, x, z][a, y, z], \quad [a, x, yz] = [a, x, y][b, x, z], \quad (4.8)$$

$$[ab, x, y] = [a, x, y][b, x, y]. \quad (4.9)$$

If G is an A -loop then

$$(ab, x) = (a, x)(b, x), (a, xy) = (a, x)(a, y), \quad (4.10)$$

$$\gamma(ab, x, y) = \gamma(a, x, y)\gamma(b, x, y),$$

$$\gamma(a, xy, z) = \gamma(a, x, z)\gamma(a, y, z), \gamma(a, x, yz) = \gamma(a, x, y)(b, x, z), \quad (4.11)$$

$$(a \setminus b, x) = (a, x)^{-1}(b, x), (a, x \setminus y) = (a, x)^{-1}(a, y), \quad (4.12)$$

$$\gamma(a \setminus b, x, y) = \gamma(a, x, y)^{-1}\gamma(b, x, y), \gamma(a, x \setminus y, z) = \gamma(a, x, z)^{-1}\gamma(a, y, z), \gamma(a, x, y \setminus z) = \gamma(a, x, y)^{-1}(b, x, z). \quad (4.13)$$

$$(a/b, x) = (a, x)(b, x)^{-1}, (a, x/y) = (a, x)(a, y)^{-1}, \quad (4.14)$$

$$\begin{aligned}\gamma(a/b, x, y) &= \gamma(a, x, y)\gamma(b, x, y)^{-1}, \gamma(a, x/y, z) = \\ \gamma(a, x, z)\gamma(a, y, z)^{-1}, \gamma(a, x, y/z) &\equiv \gamma(a, x, y)(b, x, z)^{-1},\end{aligned}\quad (4.15)$$

where $\gamma = \alpha$ or $\gamma = \beta$.

Proof. Let G be a Moufang loop. The equalities (4.5), (4.6) follow from items (iii), (iv), (1.9) of Lemma 1.4 with $x = a$ in item (i), as $[a, x] \in Z(G)$, $[[a, x], y] = 1$.

The prime equality from (4.7) follows from (1.10) with $x = a$, $y = b$ in item (i) of Lemma 1.4. Any Moufang loop is an IP -loop, then by (1.8) $(xy)^{-1} = y^{-1}x^{-1}$, $[x, y]^{-1} = [y^{-1}, x^{-1}]$, $[x, y, z]^{-1} = [z^{-1}, y^{-1}, x^{-1}]$. The centre of any loop is an abelian group. Then the second equality from (4.7) follows from item (ii) of Lemma 1.4 and (1.11) if used $(xy)^{-1} = y^{-1}x^{-1}$ and replaced z^{-1} by a , y^{-1} by x , x^{-1} by y .

By (1.11), (1.12) $L(z, a)(xy) \cdot [a^{-1}, z^{-1}] = L(z, a)x(L(z, a)y \cdot [a^{-1}, z^{-1}])$,

$$(xy)[xy, a, z]^{-1} \cdot [a^{-1}, z^{-1}] = ((x[x, a, z]^{-1})(y[y, a, z]^{-1})) \cdot [a^{-1}, z^{-1}].$$

As $[a^{-1}, z^{-1}] \in Z(G)$ then by (1.11), (1.12) $(xy)[xy, a, z]^{-1} = L(z, a)(xy) = L(z, a)x \cdot L(z, a)y = (x[x, a, z]^{-1})(y[y, a, z]^{-1})$. By (4.6) $[a, z, x], [a, z, y] \in Z(G)$ implies $[x, a, z], [y, a, z] \in Z(G)$. Then from $(xy)[xy, a, z]^{-1} = (x[x, a, z]^{-1})(y[y, a, z]^{-1})$ it follows $[xy, a, z]^{-1} = [x, a, z]^{-1}[y, a, z]^{-1}$ which by (4.6) implies (4.8).

The subloops \mathcal{Z}_2 , G_k are normal in G and $a, b \in \mathcal{Z}_2$ or $a, b \in G_k$. Then by (1.5) $L(y, x)a, L(y, x)b \in \mathcal{Z}_2$ or $L(y, x)a, L(y, x)b \in G_k$. By (1.6), diassociativity of Moufang loops and (4.6) $[x^{-1}, y^{-1}] = x^{-1}y^{-1}xy$. We use (4.7), (4.8). Then $[L(y, x)a, L(y, x)b, [x^{-1}, y^{-1}]] = [L(y, x)a, L(y, x)b, x][L(y, x)a, L(y, x)b, y][L(y, x)a, L(y, x)b, x]^{-1}[L(y, x)a, L(y, x)b, y]^{-1} = 1$. According to (1.6) $L(y, x)a(L(y, x)b \cdot [x^{-1}, y^{-1}]) = (L(y, x)a \cdot (L(y, x)b)[x^{-1}, y^{-1}])$.

By (1.12) $L(y, x)(ab)[x^{-1}, y^{-1}] = L(y, x)a(L(y, x)b \cdot [x^{-1}, y^{-1}])$. Then $L(y, x)(ab) = L(y, x)aL(y, x)b$. We use (1.11). Then $(ab)[ab, x, y] = a[a, x, y] \cdot b[b, x, y]$. Consequently, $[ab, x, y] = [a, x, y][b, x, y]$ and (4.9) is proved.

Now, let G be a centrally nilpotent of class 2 A -loop and let $a, b, x, y, z \in G$. According to (2.6) and Corollary 2.17 we assume that $a, b \in \mathcal{Z}_2(G)$. Then from (2.5) and Corollary 2.17 it follows that the commutator-associators $\alpha(a, x, y), \beta(a, x, y), (a, x)$ belong to center $Z(G)$ of loop G . By definition $ax \cdot y = a\alpha(a, x, y) \cdot xy, y \cdot xa = yx \cdot \beta(a, x, y)a, ax = x(a(a, x))$. We prove the identities (4.10) - (4.15) only for associators of type $\alpha(a, x, y)$ as for $\beta(a, x, y), (a, x)$ the corresponding identities are proved analogically.

As $\alpha(a, x, y) \in Z(G)$ then from $ax \cdot y = a\alpha(a, x, y) \cdot xy$ we get $ax \cdot y = a(\alpha(a, x, y)x \cdot y)$, $R(y)L(a)x = L(a)R(y)(\alpha(a, x, y)x)$, $S(a, y)x = \alpha(a, x, y)x$,

where $S(a, y) = R(y)^{-1}L(a)^{-1}R(y)L(a)$. Obviously, $S(a, y)1 = 1$, i.e. $S(a, y)$ is an inner mapping. Then $S(a, y)$ is an automorphism of loop G . Hence $\alpha(a, xz, y)(xz) = S(a, y)(xz) = S(a, y)x \cdot S(a, y)z = \alpha(a, x, y)x \cdot \alpha(a, z, y)z = (\alpha(a, x, y)\alpha(a, z, y))(xz)$, i.e. $\alpha(a, xz, y) = \alpha(a, x, y)\alpha(a, z, y)$. The identities $\alpha(ab, x, y) = \alpha(a, x, y)\alpha(b, x, y)$, $\alpha(a, x, yz) = \alpha(a, x, y)\alpha(a, x, z)$, $(ab, x) = (a, x)(b, x)$, $(a, xy) = (a, x)(a, y)$ are proved by analogy. Consequently, the identities (4.10), (4.11) hold for the associators of type $\alpha(x, y, z)$ and the commutators (x, y) .

Further, according to (4.10) $(a, x) = (a/b \cdot b, x) = (a/b, x)(b, x)$, $(a/b, x) = (a, x)(b, x)^{-1}$. The other identities (4.10) - (4.15) are proved in a similar manner. \square

Lemma 4.4. *Let Q be a loop with (transfinite) lower central series $\{\mathcal{A}_\xi\}$, let k be a natural number and let $a, b \in \mathcal{A}_k(Q)$, $x, y, z \in Q$.*

If Q is a Moufang loop then

$$[a, x, y] \equiv [x, y, a] \equiv [y, a, x] \pmod{\mathcal{A}_{k+2}},$$

$$[a, x, y]^{-1} \equiv [a^{-1}, x, y] \equiv [a, x^{-1}, y] \equiv [a, y, x] \pmod{\mathcal{A}_{k+2}}, \quad (4.16)$$

$$[ab, x] \equiv [a, x][b, x][a, b, x]^3 \pmod{\mathcal{A}_{k+2}},$$

$$[a, xy] \equiv [a, x][a, y][a, x, y]^3 \pmod{\mathcal{A}_{k+2}}, \quad (4.17)$$

$$[a, xy, z] \equiv [a, x, z][a, y, z] \pmod{\mathcal{A}_{k+2}},$$

$$[a, x, yz] \equiv [a, x, y][b, x, z] \pmod{\mathcal{A}_{k+2}}, \quad (4.18)$$

$$[ab, x, y] \equiv [a, x, y][b, x, y] \pmod{\mathcal{A}_{k+2}}. \quad (4.19)$$

If G is an A -loop then

$$(ab, x) \equiv (a, x)(b, x) \pmod{\mathcal{A}_{k+2}},$$

$$(a, xy) \equiv (a, x)(a, y) \pmod{\mathcal{A}_{k+2}}, \quad (4.20)$$

$$\gamma(ab, x, y) \equiv \gamma(a, x, y)\gamma(b, x, y) \pmod{\mathcal{A}_{k+2}},$$

$$\gamma(a, xy, z) \equiv \gamma(a, x, z)\gamma(a, y, z) \pmod{\mathcal{A}_{k+2}},$$

$$\gamma(a, x, yz) \equiv \gamma(a, x, y)(b, x, z) \pmod{\mathcal{A}_{k+2}}, \quad (4.21)$$

$$(a \setminus b, x) \equiv (a, x)^{-1}(b, x) \pmod{\mathcal{A}_{k+2}},$$

$$(a, x \setminus y) \equiv (a, x)^{-1}(a, y) \pmod{\mathcal{A}_{k+2}}, \quad (4.22)$$

$$\gamma(a \setminus b, x, y) \equiv \gamma(a, x, y)^{-1}\gamma(b, x, y),$$

$$\gamma(a, x \setminus y, z) \equiv \gamma(a, x, z)^{-1}\gamma(a, y, z) \pmod{\mathcal{A}_{k+2}},$$

$$\gamma(a, x, y \setminus z) \equiv \gamma(a, x, y)^{-1}(b, x, z) \pmod{\mathcal{A}_{k+2}}. \quad (4.23)$$

$$(a/b, x) \equiv (a, x)(b, x)^{-1} \pmod{\mathcal{A}_{k+2}},$$

$$(a, x/y) \equiv (a, x)(a, y)^{-1} \pmod{\mathcal{A}_{k+2}}, \quad (4.24)$$

$$\gamma(a/b, x, y) \equiv \gamma(a, x, y)\gamma(b, x, y)^{-1} \pmod{\mathcal{A}_{k+2}},$$

$$\gamma(a, x/y, z) \equiv \gamma(a, x, z)\gamma(a, y, z)^{-1} \pmod{\mathcal{A}_{k+2}},$$

$$\gamma(a, x, y/z) \equiv \gamma(a, x, y)(b, x, z)^{-1} \pmod{\mathcal{A}_{k+2}}, \quad (4.25)$$

where $\gamma = \alpha$ or $\gamma = \beta$.

Lemma 6.4 follows from Lemma 4.3 and (4.4).

Theorem 4.5. *Let the Moufang loop (respect. A-loop) Q with lower central series $\{\mathcal{A}_\alpha(Q)\}$ be generated by the set M . Then for any integer $n \geq 0$ the quotient loop $\mathcal{A}_n(Q)/\mathcal{A}_{n+1}(Q)$ is an abelian group and is generated by those cosets of Q modulo $\mathcal{A}_{n+1}(Q)$ that contain commutator-associators of type (μ) (respect. (α, β)) of weight n of the elements of M .*

Proof. We use the induction with respect to n . For $n = 0$ the statement is obvious. We assume that the quotient loop $\mathcal{A}_{n-1}(Q)/\mathcal{A}_n(Q)$ is generated by cosets of Q modulo $\mathcal{A}_n(Q)$ containing commutator-associators of type (μ) (respect. (α, β)) of weight $n - 1$ of elements of M . By Corollary 2.153 $\mathcal{A}_n(Q)$ is generated by the elements $[a, x, y]$, $[a, x]$ (respect. $\alpha(a, x, y)$, $\beta(a, x, y)$, (a, x)), where $a \in \mathcal{A}_{n-1}(Q)$, $x, y \in Q$. By induction hypothesis, $a = (a_1^{\pm 1} \dots a_r^{\pm 1}) \cdot z$, where $z \in \mathcal{A}_n$, and each a_i is a commutator-associator of type (μ) (respect. (α, β)) of weight $n - 1$ of elements of M . The elements x, y are terms with respect to variables $b_i, c_i \in M$. After using several times the identities (4.16) – (4.19) (respect. (4.20) – (4.25)), we get that $[a, x, y]$, $[a, x]$ (respect. $\alpha(a, x, y)$, $\beta(a, x, y)$, (a, x)) are products of terms of the forms $[a_i, b_i, c_i]$, $[a_i, b_i]$, $[z, x, y]$, $[z, x]$ (respect. $\alpha(a_i, b_i, c_i)$, $\beta(a_i, b_i, c_i)$, (a_i, b_i) , $\alpha(z, x, y)$, $\beta(z, x, y)$, (z, x)), where $b_i, c_i \in M$. Since $[z, x, y]$, $[z, x]$, $\alpha(z, x, y)$, $\beta(z, x, y)$, $(z, x) \in \mathcal{A}_{n+1}$ the proof end. \square

Corollary 4.6. *Any finitely generated centrally nilpotent Moufang loop or A-loop Q satisfies the maximum condition for its subloops.*

Proof. Let $Q = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \dots \mathcal{A}_n = \{1\}$ be the lower central series of the loop Q and let H be a subloop of Q . We denote $H_i = H \cap \mathcal{A}_i$. Using (1.3) it is easy to see that the subloop H_i is normal in H . By homomorphism theorems we get $H_i/H_{i+1} = H_i/(H \cap \mathcal{A}_{i+1}) \cong H_i\mathcal{A}_{i+1}/\mathcal{A}_{i+1} \subseteq \mathcal{A}_i/\mathcal{A}_{i+1}$. By Theorem 4.5 $\mathcal{A}_i/\mathcal{A}_{i+1}$ is a finitely generated abelian group. Then H_i/H_{i+1} also is finitely generated. From here it follows that the subloop H is finitely generated. \square

5 The basis of identities

Let $(F, \cdot, /, \backslash, 1)$ (respect. $(F, \cdot, ^{-1}, 1)$) be the free A -loop (respect. free Moufang loop) on a countable infinite set of free generators, g_0, g_1, g_2, \dots . Follow Evans [9]. For A -loops we use the designation (3.3), $x^{-1} = 1/x = x \backslash 1$. A *simple associator* in A -loop (respect. Moufang loop) F is defined as follows. Each g_i and g_i^{-1} is a simple associator. If u, v, w are simple associators in A -loop (respect. Moufang loop) F , then so is $\alpha(u, v, w), \beta(u, v, w)$ (respect. $[u, v, w]$).

A simple associator in F is said to *involve* the generator g_i if (i) it is g_i or g_i^{-1} , (ii) if it is $\alpha(u, v, w), \beta(u, v, w)$ for A -loop and $[u, v, w]$ for Moufang loop where at least one of u, v, w involves g_i .

Note that from the definitions of associators $\alpha(a, b, c), \beta(a, b, c), [a, b, c]$ and (1.4) it follows that if a simple associator involves g_i , then it lies in the normal subloop of $(F, \cdot, 1)$ generated by g_i .

For $i = 0, 1, 2, \dots$ we define the endomorphisms δ_i , of F given by $\delta_i g_i = 1$, $\delta_i g_j = g_j$ for $i \neq j$. The kernel of δ_i is the normal subloop of F generated by g_i .

According to item 8) in Section 3 let $\{F_\xi\}$ be the lower central series of loop $(F, \cdot, 1)$ (or, that same, of loop $(F, \cdot, /, \backslash, 1)$). Follow Evans [9, Lemma 2].

Lemma 5.1. *If w is an element in F and w maps onto 1 under the endomorphisms δ_i , $i = 0, 1, 2, \dots, 2n$, then $w \in F_n$ can be written as a product of simple associators each of which involves g_0, g_1, \dots, g_{2n} and $w \in F_n$.*

Proof. We use the Propositions 3.2, 3.3 and item 8) of Section 3.

We prove the assertion of Lemma 5.1 by induction on n . For $n = 0$ we prove this for one generator g_0 . The kernel of δ_0 is the normal subloop in F generated by g_0 . According to Lemma 1.1 $\ker \delta_0$ is generated as a subloop by set $\{\varphi g_0 \mid \varphi \in \mathfrak{I}(Q)\}$, where $\mathfrak{I}(Q)$ is the inner mapping group of Q . By (1.11) and (2.3)

$$L(z.y)x = x[x, y, z]^{-1}, \quad R(b, c)a = a\alpha(a, b, c), L(c, b)a = \beta(a, b, c)a. \quad (5.1)$$

The property of being a product of simple associators involving g_0 is clearly preserved under multiplication. Furthermore, from (5.1) it follows that for the proof of Lemma 5.1 at $i = 0$ is sufficient to prove that the application of any inner mappings $R(u, v), L(u, v)$ to a product of simple associators involving g_0 also preserves this property. For A -loops it follows from definition of A -loop.

Let F be a Moufang loop. We define $x \equiv 1$ if $x \in \ker \delta_0$. From $x, y \in \ker \delta_0$ it follows $[x, u, v] \equiv 1$, $[y, u, v] \equiv 1$ for all $u, v \in F$ by (5.1). It is necessary to show that $[xy, u, v] \equiv 1$. But it follows from Lemmas 1.2, 1.3.

Clearly, that $\ker \delta_0 \subseteq F_0 = F$. Hence the lemma is proved for $n = 0$.

We suppose by the induction hypothesis that lemma is correct for $n - 1$. By Theorem 4.5 F_{n-1} modulo F_n is generated by commutator-associator of the type (α, β) (or commutator-associator of the type (μ)) of weight $n - 1$ of the variables g_1, g_2, \dots . Then

$$w \cong \prod_{i \in I} v_i^{\pm 1} \pmod{F_n}, \quad (5.2)$$

where the set I is finite and for all $i \in I$ the element v_i is a commutator-associator of weight $n - 1$. As every commutator-associator of weight $n - 1$ contains at most $2n - 2$ variables, each v_i , $i \in I$, does not contain some variable of the set $\{g_1, g_2, \dots, g_{2n}\}$.

Let the commutator-associator v_j from (5.2) contains the variables $g_{j_1}, g_{j_2}, \dots, g_{j_r}$. If $i \in \{j_1, j_2, \dots, j_r\}$, then $\delta_i v_j = 1$, in the contrary case $\delta_i v_j = v_j$. Further, from Corollary 2.17 it follows that the subloop F_n is fully invariant, particularly, $\delta_i F_n \subseteq F_n$. To apply the endomorphism δ_0 at (5.2). We obtain

$$1 \cong \delta_i w \cong \prod_{i \in I_i} v_i^{\pm 1} \pmod{F_{n+1}},$$

where the subset I_0 of $I = \{0, 1, 2, \dots, 2n\}$ consists of those i for which v_i does not contain g_0 . Hence we can rewrite w as

$$w \cong \prod_{i \in I \setminus I_0} v_i^{\pm 1} \pmod{F_n}.$$

If $I = I_0$ then $w \in F_n$ and the proof is complete. If $I \neq I_0$, then we repeat the procedure above for the representation of w and the endomorphism δ_1 . In at most $2n$ steps we will obtain that $w \in F_n$. \square

Lemma 5.2. *Let w be a word in F in g_0, g_1, \dots, g_t where $t \geq 2n$. Then*

$$w = (\dots (uv_0)v_1 \dots)v_{t-1})v_t,$$

where $u \in F_n$ and each v_i is a product of words of the form $\delta_{i_s} \dots \delta_{i_2} \delta_{i_1} w^{\pm 1}$, for some nonempty sequence i_1, i_2, \dots, i_s from $0, 1, 2, \dots, t$.

Proof. We follow Evans [9, Lemma 3]. Define γ_i , $i = 0, 1, 2, \dots$ on words in F by $\gamma_i w = w \cdot (\delta_i w)^{-1}$. If F is the A -loop then $\gamma_i w = w \cdot (\delta_i w)^{-1} =$

$w \cdot (1/\delta_i w) = w \cdot \delta_i(1/w) = w \cdot \delta_i w^{-1}$. Similarly $\delta_i w = w \cdot \delta_i w^{-1}$ for Moufang loop F .

Let $u = \gamma_t \dots \gamma_1 \gamma_0 w$. Then $\delta_i w = 1$, for $i = 0, 1, 2, \dots, t$. By Lemma 5.1, $u \in F_n$. Now $\gamma_0 w = w \cdot \delta_0 w^{-1}$, $\gamma_1 \gamma_0 w = (w \cdot \delta_0 w^{-1})(\delta_1 \delta_0 w \cdot \delta_1 w^{-1})$, and in general $\gamma_t \dots \gamma_1 \gamma_0 w = (\dots (w w_0) w_1 \dots) w_t$, where each w_i is an expression of words of the form $\delta_{i_s} \dots \delta_{i_2} \delta_{i_1} w^{\pm 1}$ for some nonempty sequence i_1, i_2, \dots, i_s from $0, 1, 2, \dots, t$. Hence $w = (\dots ((u v_0) v_1) \dots) v_t$, where $v_0 = w_t^{-1}$, $v_1 = w_{t-1}^{-1}$, \dots , $v_t = w_0^{-1}$. \square

Lemma 5.3. *Any identity $w(x_1, x_2, \dots) = 1$ in a Moufang loop or A -loop defined by three basic binary operations is equivalent to a finite collection of identities $w_i = 1$, $i = 1, 2, 3, \dots$, where some of the w_i belong to F_n and the remainder involve at most $2n$ variables.*

Proof. If $w = 1$ involves fewer than $2n$ variables, there is nothing to prove. If this is not the case, we use Lemma 5.2, writing $w = 1$ as

$$((\cdot(u \cdot v_1) \cdot v_2) \cdot \dots) \cdot w_s = 1.$$

where v_i are products of words in x_1, x_2, \dots of the form $\delta_{i_s} \dots \delta_{i_2} \delta_{i_1} w^{\pm 1}$. It follows that $w = 1$ is equivalent to $u = 1$ where $u \in F_n$, and identities $\delta_{i_s} \dots \delta_{i_2} \delta_{i_1} w = 1$, since each of the $\delta_{i_s} \dots \delta_{i_2} \delta_{i_1} w = 1$ is a consequence of $w = 1$. Now the $\delta_{i_s} \dots \delta_{i_2} \delta_{i_1} w = 1$ all involve at least one less variable than $w = 1$. We may repeat the process with these new identities until we obtain, in addition to the identities in F_n , identities involving at most $2n$ variables, as required. \square

Theorem 5.4. *The identities of a centrally nilpotent Moufang loop (respect. A -loop) of class n are finitely based. The basis consists from one associator identity of type (μ) (respect. of type (α) or (β)) of weight n in $2n + 1$ variables and a finite collection of identities in no more than $2n$ variables.*

Proof. We use the Theorem 3.4. Let Q be a centrally nilpotent Moufang loop or A -loop of class n and let $(F, \cdot, /, \backslash, 1)$ be the free Moufang loop or free A -lop on a countable infinite set of free generators, g_1, g_2, \dots . Let H be the subloop of F generated by $g_1, g_2, \dots, g_{2n+1}$ and W the word subloop generated by the identities of Q .

By Lemma 5.3, W is generated, as a fully invariant subloop of F , by words in g_1, g_2, \dots, g_{2n} and words in F_n . Since Q is centrally nilpotent of class n , $W \supseteq F_n$ and so W is generated, as a fully invariant subloop, by set of words in g_1, \dots, g_{2n} and the single associator word of type (μ) (respect. (α) or (β)) of weight n in $2n + 1$ variables. By Corollary 4.6, the subloop of H generated by these words can be generated by a finite set of words. This

finite set of words generates W as a fully invariant subloop of F . Hence, F/W is finitely based and so are the identities of Q . \square

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